Lecture 4 (1/10/22)

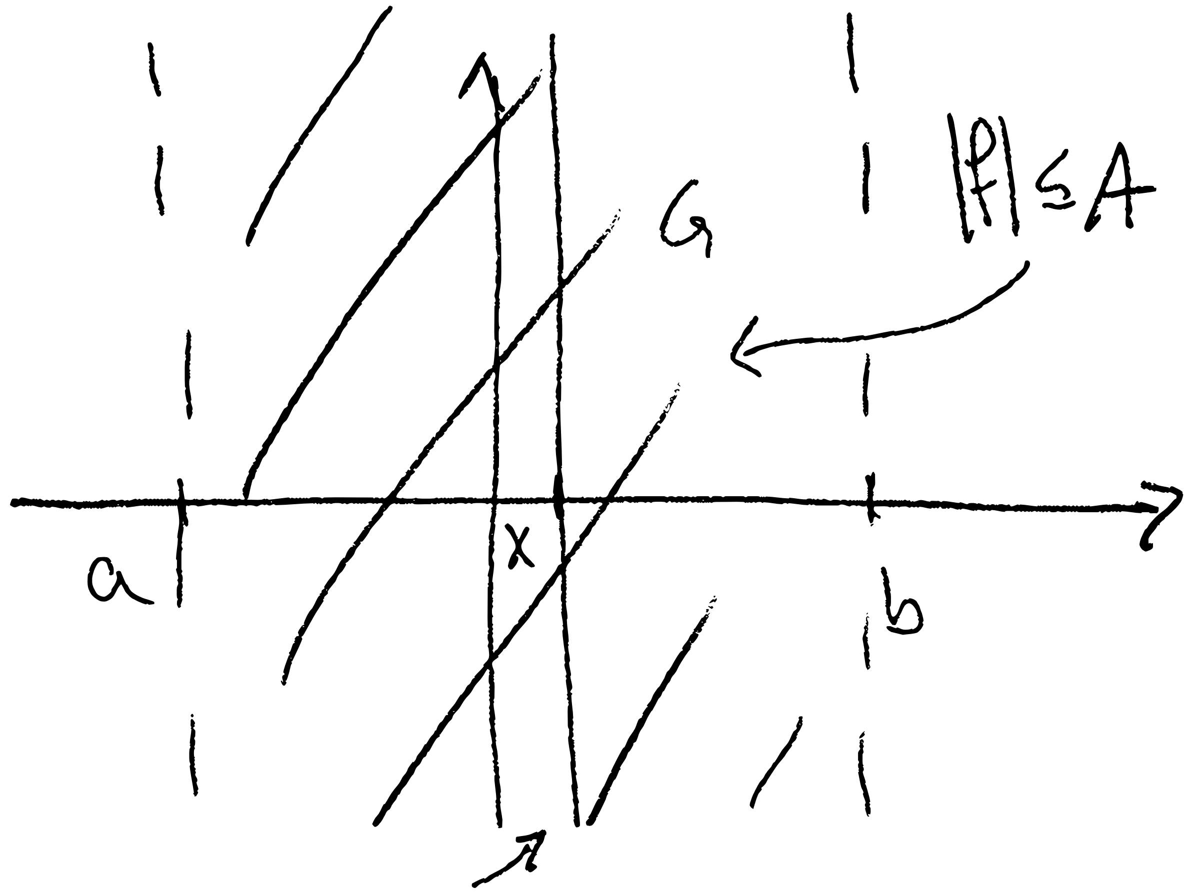
We are in the middle of proving the following Max Mod Princ. / convexity result.

Thm 1. Let $G = \{a < \operatorname{Re} z < b\}$ and f analytic in G, cont. in $\overline{G}$, and $|f| \leq A$. Let $M(x) := \sup_{y \in \mathbb{R}} |f(x+iy)|$

If $f \not\equiv 0$, then $x \to M(x)$ is log-convex.

Pf. Recall that we proved that $f \not\equiv 0 \Rightarrow M(x) > 0$, which is needed to talk about log-convexity. ($f \not\equiv 0 \Rightarrow M(x) > 0$ was part (i) of thm proved last lecture. Thm 1 above was part (ii) of thm stated in last lecture.)

For the pf of Thm 1, we shall need the following lemma, where f, G are as above:



$$M(x) = \sup_{y \in \mathbb{R}} |f(x+iy)|$$

<u>Lemma 1.</u> Assume $M(a) \le 1$, $M(b) \le 1$, then $|f(z)| \le 1$.

<u>Rem.</u> ① Condition $|f| \le A$ in G is needed here by ~~Ex~~: $b = -a = \pi/2$, $f(z) = e^{e^{iz}}$ (cf. Ex in Lecture 1).

② One should compare Lemma 1 to MMT-III. Note no condition is needed at $\infty \in \partial_\infty G$ here. The geometry of G allows for $|f| \le A$ to replace this condition.

Pf of Lemma 1. Consider, for $\varepsilon > 0$,

$$g_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}, \quad z \in G.$$

Note: $|g(z)| \le \min\left(\frac{1}{|\mathrm{Re}(1+\varepsilon(z-a))|}, \frac{1}{|\mathrm{Im}(1+\varepsilon(z-a))|}\right)$

$$= \min\left(\frac{1}{1+\varepsilon(x-a)}, \frac{1}{\varepsilon|y|}\right), \quad z = x+iy \in G.$$

Then, fg_ε is analytic in G, cont. in $\overline{G}$. Moreover, $|f(z)g_\varepsilon(z)| \le |f(z)| \Rightarrow$ $|fg_\varepsilon| \le 1$ on ∂G (since $M(a), M(b) \le 1$), and $\limsup_{z\to\infty} |f(z)g_\varepsilon(z)| \le \limsup_{\substack{z\to\infty \\ a<\mathrm{Re}\, z<b}} \frac{A}{\varepsilon|y|}$

$= 0 \le 1$. By MMT-III, $|fg_\varepsilon| \le 1$ in G. Thus, for any fixed $z \in G$, $|f(z)| \le |1+\varepsilon(z-a)|$ for every $\varepsilon > 0$. Letting $\varepsilon \to 0 \Rightarrow |f(z)| \le 1$. ☒

Rem. The same conclusion would hold if we only assume $|f(z)| \le A\,|\mathrm{Im}\, z|^{p}$ for some p. DIY.

Pf of Thm 1, contd. Log-convexity of $M(x)$ means, for all $a \le x_1 < x < x_2 \le b$,

$$\log M(x) \le \frac{x-x_1}{x_2-x_1}\log M(x_2) + \frac{x_2-x}{x_2-x_1}\log M(x_1)$$

or, $\Leftrightarrow$

$$M(x) \le M(x_2)^{\frac{x-x_1}{x_2-x_1}}\, M(x_1)^{\frac{x_2-x}{x_2-x_1}} \qquad (*)$$

A moment's reflection $\Rightarrow$ suffices to consider $x_1 = a$, $x_2 = b$.

Consider $\quad g(z) = M(b)^{\frac{z-a}{b-a}}\, M(a)^{\frac{b-z}{b-a}}$,

where for a given $B > 0$ and $w \in \mathbb{C}$, $B^{w} = e^{w \log B}$. Thus, g is entire and

$$|g(z)| = M(b)^{\frac{x-a}{b-a}}\, M(a)^{\frac{b-x}{b-a}}$$

since $|B^{w}| = B^{\mathrm{Re}\, w}$.

Note, $|g(z)|$ is >0 function of x in $[a,b]$ for $z \in \overline{G}$ $\Rightarrow$ $|g(z)| \geq \delta > 0$ in $\overline{G}$. Consider then the analytic in G, cont. in $\overline{G}$ function $h = f/g$.

We have $|h| \leq A/\delta$ in G, and

$$|h(a+iy)| \leq \frac{M(a)}{M(a)} = 1, \quad |h(b+iy)| \leq \frac{M(b)}{M(b)} \leq 1$$

so by Lemma 1, $|h| \leq 1$ in G. But

$$\Rightarrow |f(x+iy)| \leq |g(z)| = M(b)^{\frac{x-a}{b-a}} M(a)^{\frac{b-x}{b-a}}.$$

Since $M(x) = \sup_{y \in \mathbb{R}} |f(x+iy)|$ $\Rightarrow$ $(*)$ w/

$x_1 = a$, $x_2 = b$ as desired. $\boxtimes$

A generalization of Thm 1.

Recall that we noted in pf of Lemma 1 that we can replace $|f| \le A$ by $|f(z)| \le A |\operatorname{Im} z|^p$ for some $p > 0$.

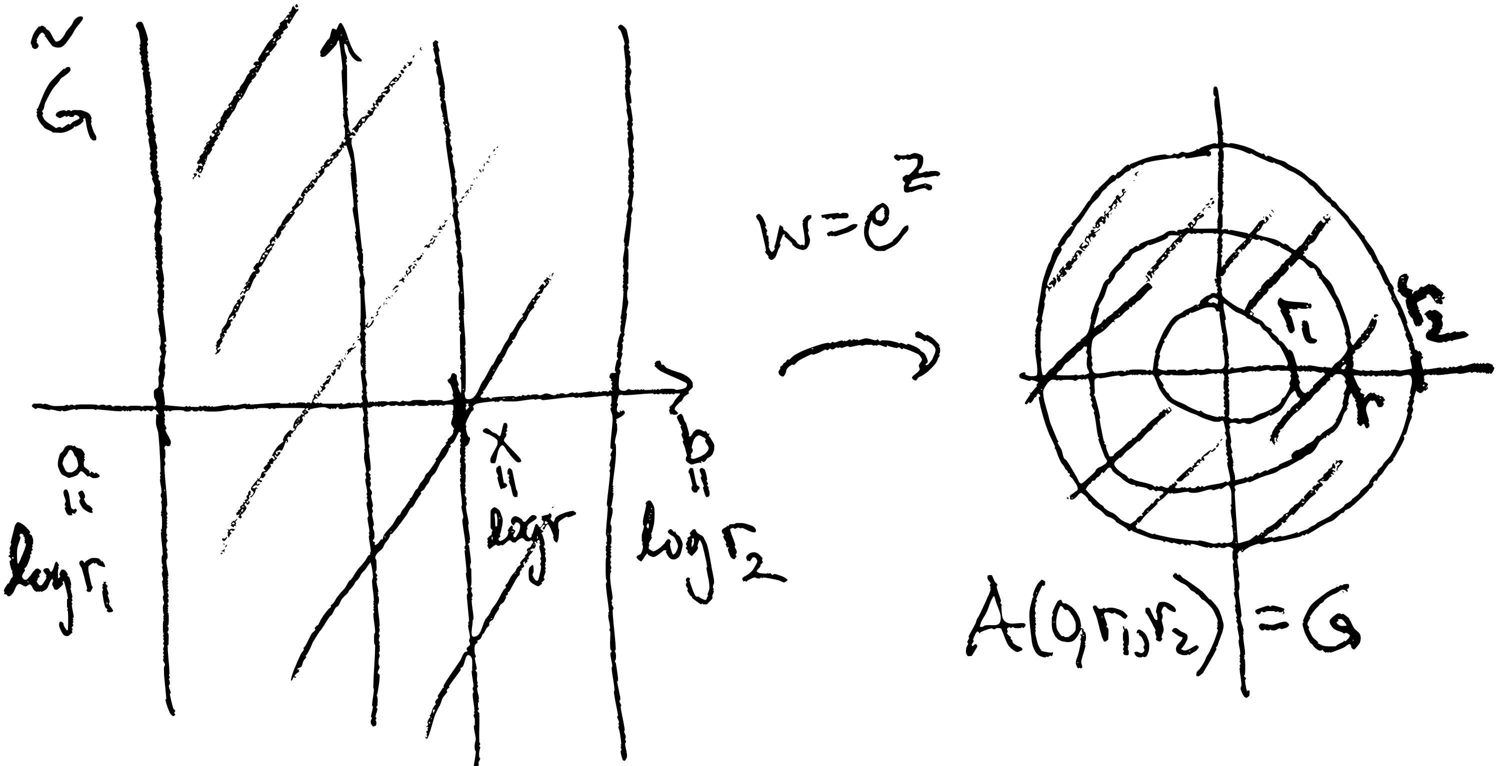
We simply replace $g_\varepsilon(z) = \dfrac{1}{1+\varepsilon(z-a)}$

by $g_\varepsilon(z) = \dfrac{1}{(1+\varepsilon(z-a))^q}$, where

q is an integer, $q > p$, in pf. Using so generalized Lemma 1, we obtain a generalized Thm 1: For G as in Thm 1, If $|f(z)| \le A |\operatorname{Im} z|^p$ and $M(b), M(a) < \infty$, then $M(x) < \infty$ and $x \to M(x)$ is log-convex.

Hadamard's 3 circles Thm.

Consider the following conformal map:



If f is analytic in $G = A(0, r_1, r_2)$ and cont. on $\overline{G}$, then $\tilde{f}(z) = f(e^z)$ is anal. in $\tilde{G}$, cont. on $\overline{\tilde{G}}$, and $|\tilde{f}| \le A$.

If we set $M(r) = \sup_{|z|=r} |f(z)| =$

$$= \sup_{y \in \mathbb{R}} |\tilde{f}(x+iy)| = \tilde{M}(x), \quad x = \log r.$$

Thm 1 states that $\log \tilde{M}(x)$ is cvx fcn of $x = \log r \Rightarrow$

$\log M(r)$ is cvx. function of $\log r$.

This is <u>Hadamards 3 circles Thm</u>:

Let f be analytic in $G = A(0, r_1, r_2)$, cont. in $\overline{G}$, and let $M(r) = \sup_{|z|=r} |f(z)|$.

Then $\log M(r)$ is convex fcn of $\log r$:

$$\log M(r) \le \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2) + \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1).$$

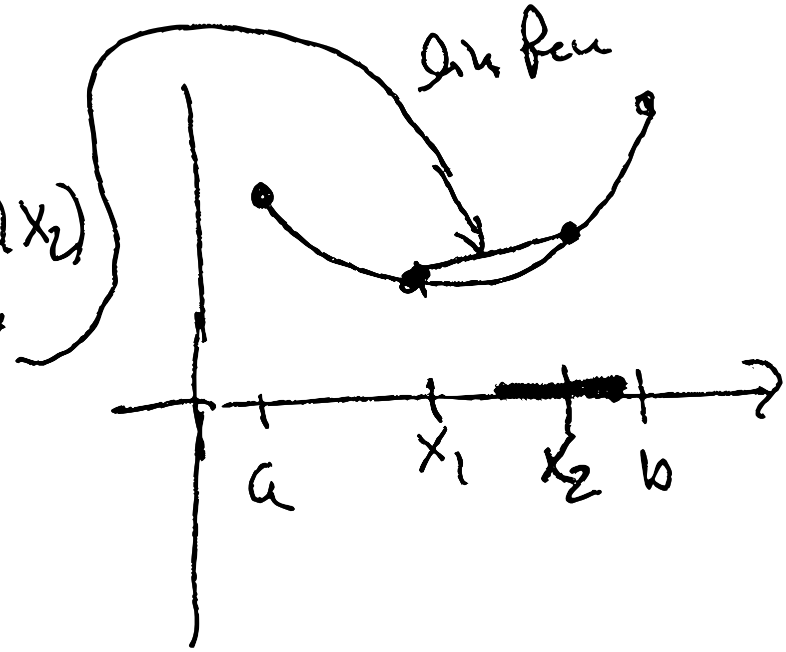
or

$$\log \frac{r_2}{r_1} \log M(r) \le \log \frac{r}{r_1} \log M(r_2) + \log \frac{r_2}{r} \log M(r_1).$$

Appendix. Log-convexity vs convexity.

Convexity:

$$u(x) \leq \frac{x-x_1}{x_2-x_1} u(x_2) + \frac{x_2-x}{x_2-x} u(x_2)$$



Log-convexity. $(u>0)$

$$u(x) \leq u(x_2)^{\frac{x-x_1}{x_2-x_1}} \underline{\underline{u(x_1)^{\frac{x_2-x}{x_2-x_1}}}}$$